

Justification Logic of the Lambda Calculus

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Introduction to justification logic

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- ▶ $\Box A \rightsquigarrow [t]A$, where t is a **proof** of A , or explicit knowledge of A .
- ▶ Read this as t **justifies** A .

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$$LP \vdash [t]A \iff PA \vdash \text{Proof}(t, A^*)$$

- ▶ The modal logic S4 is an informal **classical provability semantics** for intuitionistic propositional logic (IPL).
- ▶ Proof terms refer to **intuitionistic proofs**.

Internalisation of justification logic

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Lemma (Lifting Lemma)

If $A_1, \dots, A_n \vdash_{\text{LP}} B$, then there exists a proof term $t(x_1, \dots, x_n)$ such that

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This mirrors the **external** modal view:

$$\text{k} \frac{A_1, \dots, A_n \vdash B}{\Box A_1, \dots, \Box A_n \vdash \Box B}$$

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- ▶ Base axiomatisation extends CPL with
 - jk : $[s](A \rightarrow B) \rightarrow ([t]A \rightarrow [s \cdot t]B)$
 - jt : $[t]A \rightarrow A$
 - j4 : $[t]A \rightarrow [!t][t]A$
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jk : $[s](A \rightarrow B) \rightarrow ([t]A \rightarrow [s \cdot t]B)$

$$\text{jt} : [t]A \rightarrow A$$

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$$\text{j+} : ([s]A \vee [t]A) \rightarrow [s + t]A$$

$$\text{can} \frac{A \text{ is an axiom instance}}{[c_n][\dots][c_1]A}$$

- ▶ Connection from combinatory logic to LP

[Shamkanov]

$$\frac{A \text{ is an axiom instance}}{c:A}$$

$$\frac{M:A \rightarrow B \quad N:A}{MN:B}$$

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What we do:

- ▶ Construct the logic axiomatically, explore internalisation features.
- ▶ Explore the logic's computation through a **Curry-Howard** interpretation.
- ▶ Establish a connection between proof terms and the term calculus.
- ▶ Achieve **normalisation** through **cut-elimination** in a sequent calculus.

Logic J_λ

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- Proof terms are that of the simply typed λ -calculus extended with !

$$t ::= x \mid t \cdot t \mid \lambda x. t \mid \pi_l(t) \mid \pi_r(t) \mid \langle t, t \rangle \mid !t$$

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- ▶ Define **closed** terms as standard.
- ▶ *Formulas* of the language $\mathcal{L}_J$ are given by the grammar

$$A ::= p \mid (A \wedge A) \mid (A \rightarrow A) \mid [t]A$$

- ▶ NB: working in the negative fragment.

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$$j_{\rightarrow_I} : ([x]A \rightarrow [t]B) \rightarrow [\lambda x.t](A \rightarrow B)$$

$$j_{\wedge_I} : [s]A \rightarrow ([t]B \rightarrow [\langle s, t \rangle](A \wedge B))$$

$$j_{\wedge_{E_l}} : [t](A \wedge B) \rightarrow [\pi_l(t)]A$$

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- Consecutions $\Gamma \vdash_{J_\lambda} A$ where $\Gamma \subseteq \mathcal{L}_J$ and $A \in \mathcal{L}_J$ is defined inductively as follows:

$$\frac{A \text{ is an axiom instance}}{\Gamma \vdash_{J_\lambda} A} \quad \frac{A \in \Gamma}{\Gamma \vdash_{J_\lambda} A} \quad \text{mp} \frac{\Gamma \vdash_{J_\lambda} A \quad \Gamma \vdash_{J_\lambda} A \rightarrow B}{\Gamma \vdash_{J_\lambda} B}$$

- We say A is a *theorem* of J_λ if $\vdash_{J_\lambda} A$.

Embedding of λ -calculus

Theorem

Let $A_1, \dots, A_n, A \in \mathcal{L}_{\text{IPL}}$ and $x_1, \dots, x_n \in \text{PVar}$. Then for some proof term $t(x_1, \dots, x_n)$,

$$x_1:A_1, \dots, x_n:A_n \vdash_{\lambda\text{IPL}} t:A \implies [x_1]A_1, \dots, [x_n]A_n \vdash_{J_\lambda} [t]A$$

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Proof.

Proceed by induction on $x_1:A_1, \dots, x_n:A_n \vdash_{\lambda\text{IPL}} t:A$. □

Internalisation of J_λ

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Theorem (Self-internalisation)

If $\vdash_{J_\lambda} A$, then there exists a closed proof term t such that $\vdash_{J_\lambda} [t]A$.

Proof.

Proceed by induction on $\vdash_{J_\lambda} A$.



Natural Deduction

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- ▶ The term calculus, denoted as the set $\lambda P\text{Terms}$ is defined by the following grammar:

$$\begin{aligned} M ::= & a \mid MM \mid \lambda a.M \mid \pi_l(M) \mid \pi_r(M) \mid \langle M, M \rangle \\ & \mid P_x(M) \mid !M \mid A^\square(M, M) \mid \lambda^\square a.M \mid \pi_l^\square(M) \mid \pi_r^\square(M) \mid \langle M, M \rangle^\square \mid U(M) \end{aligned}$$

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- ▶ **Typed judgements** written $\vec{\Gamma} \vdash_{\lambda J_\lambda} M:A$
- ▶ $\vec{\Gamma} = \{a_1:A_1, \dots, a_n:A_n\}$

Natural Deduction System NJ_λ

$$\begin{array}{c}
 \text{id} \frac{}{\vec{\Gamma}, a:A \vdash_{\lambda J_\lambda} a:A} \\
 \\
 \rightarrow_I \frac{\vec{\Gamma}, a:A \vdash_{\lambda J_\lambda} M:B}{\vec{\Gamma} \vdash_{\lambda J_\lambda} \lambda a.M:A \rightarrow B} \qquad \rightarrow_E \frac{\vec{\Gamma} \vdash_{\lambda J_\lambda} M:A \rightarrow B \quad \vec{\Gamma} \vdash_{\lambda J_\lambda} N:A}{\vec{\Gamma} \vdash_{\lambda J_\lambda} MN:B} \\
 \\
 \wedge_I \frac{\vec{\Gamma} \vdash_{\lambda J_\lambda} M:A \quad \vec{\Gamma} \vdash_{\lambda J_\lambda} N:B}{\vec{\Gamma} \vdash_{\lambda J_\lambda} \langle M, N \rangle : A \wedge B} \\
 \\
 \wedge_{E_l} \frac{\vec{\Gamma} \vdash_{\lambda J_\lambda} M:A \wedge B}{\vec{\Gamma} \vdash_{\lambda J_\lambda} \pi_l(M):A} \qquad \wedge_{E_r} \frac{\vec{\Gamma} \vdash_{\lambda J_\lambda} M:A \wedge B}{\vec{\Gamma} \vdash_{\lambda J_\lambda} \pi_r(M):B}
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$$\begin{array}{c}
 \begin{array}{c} \vec{\Gamma} \vdash_{\lambda J_\lambda} M:A \\ \hline [\]_I \vec{\Gamma} \vdash_{\lambda J_\lambda} P_x(M):[x]A \end{array}
 \quad
 \begin{array}{c} \vec{\Gamma} \vdash_{\lambda J_\lambda} M:[t]A \\ \hline [!]_I \vec{\Gamma} \vdash_{\lambda J_\lambda} !M:[!t][t]A \end{array}
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 \\
 \\
 \begin{array}{c} \vec{\Gamma} \vdash_{\lambda J_\lambda} M:[s](A \rightarrow B) \quad \vec{\Gamma} \vdash_{\lambda J_\lambda} N:[t]A \\ \hline [\cdot]_I \vec{\Gamma} \vdash_{\lambda J_\lambda} A^\square(M, N):[s \cdot t]B \end{array}
 \\
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 \begin{array}{c} \vec{\Gamma}, a:[x]A \vdash_{\lambda J_\lambda} M:[t]B \\ \hline [\lambda x.]_I \vec{\Gamma} \vdash_{\lambda J_\lambda} \lambda^\square a.M:[\lambda x.t](A \rightarrow B) \end{array}
 \\
 \\
 \begin{array}{c} \vec{\Gamma} \vdash_{\lambda J_\lambda} M:[t](A \wedge B) \\ \hline [\pi_l()]_I \vec{\Gamma} \vdash_{\lambda J_\lambda} \pi_l^\square(M):[\pi_l(t)]A \end{array}
 \quad
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 \\
 \\
 \begin{array}{c} \vec{\Gamma} \vdash_{\lambda J_\lambda} M:[s]A \quad \vec{\Gamma} \vdash_{\lambda J_\lambda} N:[t]B \\ \hline [\langle, \rangle]_I \vec{\Gamma} \vdash_{\lambda J_\lambda} \langle M, N \rangle^\square:[\langle s, t \rangle](A \wedge B) \end{array}
 \end{array}$$

Curry-Howard Correspondence

Theorem

$$\Gamma \vdash_{J_\lambda} A \iff \vec{\Gamma} \vdash_{\lambda J_\lambda} M:A$$

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If $\vdash_{J_\lambda} A$, then there exists a closed proof term t such that $\vdash_{J_\lambda} [t]A$ and $\vdash_{\lambda J_\lambda} t^ : A$.*

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If $a_{A_1}:A_1, \dots, a_{A_n}:A_n \vdash_{\lambda J_\lambda} M(a_{A_1}, \dots, a_{A_n}):A$ then we have $[a_{A_1}^]A_1, \dots, [a_{A_n}^*]A_n \vdash_{J_\lambda} [M(a_{A_1}, \dots, a_{A_n})^*]A$.*

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- ▶ This is a feature LP does not have with its C-H as LP cannot express λ -abstraction.

Normalisation

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- ▶ With additional justification rules any interactions between $[?]_I$ and $[]_E$ additionally needs to be removed.

Normalisation

$$\begin{array}{c}
 \overrightarrow{\Gamma} \vdash_{\lambda J_{\lambda}} M:[t]A \\
 \hline
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 \hline
 []\text{E} \quad \overrightarrow{\Gamma} \vdash_{\lambda J_{\lambda}} U(!M):[t]A
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 \rightsquigarrow
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- ▶ Easier to do this through sequent calculus and **cut-elimination** as we are working in the negative fragment.

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- ▶ A *sequent* is an ordered pair $\Gamma \Rightarrow A$ where Γ is a **multiset** of formulas.

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- ▶ A *sequent* is an ordered pair $\Gamma \Rightarrow A$ where Γ is a **multiset** of formulas.
- ▶ Sequent $\Gamma \Rightarrow A$ interpreted as

$$\text{fm}(\Gamma \Rightarrow A) = \bigwedge \Gamma \rightarrow A$$

Sequent calculus LJ_λ

$$\begin{array}{c} \text{id} \frac{}{\Gamma, p \Rightarrow p} \quad \text{id} \frac{}{\Gamma, [t]A \Rightarrow [t]A} \quad \text{c} \frac{\Gamma, A, A \Rightarrow B}{\Gamma, A \Rightarrow B} \\[10pt] \wedge_l \frac{\Gamma, A, B \Rightarrow C}{\Gamma, A \wedge B \Rightarrow C} \quad \wedge_r \frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B} \\[10pt] \rightarrow_l \frac{\Gamma, B \Rightarrow C \quad \Gamma \Rightarrow A}{\Gamma, A \rightarrow B \Rightarrow C} \quad \rightarrow_r \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \rightarrow B} \end{array}$$

Sequent calculus LJ_λ

$$\begin{array}{c}
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 [\cdot]_l \frac{\Gamma, A \Rightarrow B}{\Gamma, [t]A \Rightarrow B} \quad
 [\cdot]_r \frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow [x]A} \quad
 [!]_r \frac{\Gamma \Rightarrow [t]A}{\Gamma \Rightarrow [!t][t]A} \\
 [\cdot]_r \frac{\Gamma \Rightarrow [s](A \rightarrow B) \quad \Gamma \Rightarrow [t]A}{\Gamma \Rightarrow [s \cdot t]B} \quad
 [\lambda x. \cdot]_r \frac{\Gamma, [x]A \Rightarrow [t]B}{\Gamma \Rightarrow [\lambda x. t](A \rightarrow B)} \\
 [\pi_l()]_r \frac{\Gamma \Rightarrow [t](A \wedge B)}{\Gamma \Rightarrow [\pi_l(t)]A} \quad
 [\pi_r()]_r \frac{\Gamma \Rightarrow [t](A \wedge B)}{\Gamma \Rightarrow [\pi_r(t)]B} \\
 [\langle, \rangle]_r \frac{\Gamma \Rightarrow [s]A \quad \Gamma \Rightarrow [t]B}{\Gamma \Rightarrow [\langle s, t \rangle](A \wedge B)}
 \end{array}
 \end{array}$$

Sequent calculus LJ_λ

Theorem

The following are equivalent:

1. $\vdash_{\text{J}_\lambda} A$;
2. $\vdash_{\text{LJ}_\lambda + \text{cut}} \Rightarrow A$;
3. $\vdash_{\text{LJ}_\lambda} \Rightarrow A$.

where the cut rule is

$$\text{cut} \frac{\Gamma \Rightarrow A \quad \Delta, A, \dots, A \Rightarrow B}{\Gamma, \Delta \Rightarrow B}$$

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Proof.

- 1 $\implies$ 2 is a standard completeness argument,
2 $\implies$ 3 follows from a cut-elimination argument and
3 $\implies$ 1 follows by showing soundness of the rules in LJ_λ . □

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- ▶ Instead the rank of a cut-formula depends on the the derivations preceeding it.

$$\begin{array}{c} \begin{array}{c} \triangle \\ \pi_1 \end{array} \quad \begin{array}{c} \triangle \\ \pi_2 \end{array} \\ \frac{[\cdot]_r \frac{\Gamma \Rightarrow [s](A \rightarrow B) \quad \Gamma \Rightarrow [t]A}{\Gamma \Rightarrow [s \cdot t]B}}{\Gamma \Rightarrow [s \cdot t]B} \end{array}$$

$$\begin{aligned} & \text{rank}(\Gamma \Rightarrow [s \cdot t]B^\downarrow, \pi) \\ &= 1 + \text{rank}(\Gamma \Rightarrow [s](A \rightarrow B)^\downarrow, \pi_1) + \text{rank}(\Gamma \Rightarrow [t]A^\downarrow, \pi_2). \end{aligned}$$

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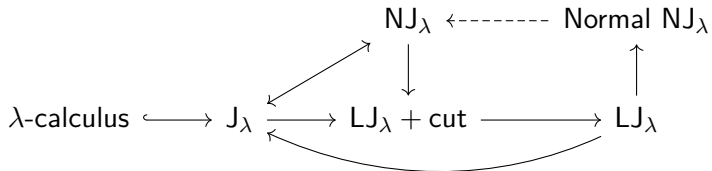
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- ▶ Intro rules correspond to right rules, elimination rules correspond to using cut.
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- ▶ Cut-free proofs are then translated to **normal** proofs.

Summary of results



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Thank you! Questions?